

ON HALF-LINE SPECTRA FOR A CLASS OF NON-SELF-ADJOINT HILL OPERATORS

KWANG C. SHIN

ABSTRACT. In 1980, Gasymov showed that non-self-adjoint Hill operators with complex-valued periodic potentials of the type $V(x) = \sum_{k=1}^{\infty} a_k e^{ikx}$, with $\sum_{k=1}^{\infty} |a_k| < \infty$, have spectra $[0, \infty)$. In this note, we provide an alternative and elementary proof of this result.

1. INTRODUCTION

We study the Schrödinger equation

$$(1) \quad -\psi''(z, x) + V(x)\psi(z, x) = z\psi(z, x), \quad x \in \mathbb{R},$$

where $z \in \mathbb{C}$ and $V \in L^\infty(\mathbb{R})$ is a continuous complex-valued periodic function of period 2π , that is, $V(x + 2\pi) = V(x)$ for all $x \in \mathbb{R}$. The *Hill operator* H in $L^2(\mathbb{R})$ associated with (1) is defined by

$$(Hf)(x) = -f''(x) + V(x)f(x), \quad f \in W^{2,2}(\mathbb{R}),$$

where $W^{2,2}(\mathbb{R})$ denotes the usual Sobolev space. Then H is a densely defined closed operator in $L^2(\mathbb{R})$ (see, e.g., [2, Chap. 5]).

The spectrum of H is purely continuous and a union of countably many analytic arcs in the complex plane [9]. In general it is not easy to explicitly determine the spectrum of H with specific potentials. However, in 1980, Gasymov [3] proved the following remarkable result:

Theorem 1 ([3]). *Let $V(x) = \sum_{k=1}^{\infty} a_k e^{ikx}$ with $\{a_k\}_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$. Then the spectrum of the associated Hill operator H is purely continuous and equals $[0, \infty)$.*

In this note we provide an alternative and elementary proof of this result. Gasymov [3] proved the existence of a solution ψ of (1) of the form

$$\psi(z, x) = e^{i\sqrt{z}x} \left(1 + \sum_{j=1}^{\infty} \frac{1}{j + 2\sqrt{z}} \sum_{k=j}^{\infty} \nu_{j,k} e^{ikx} \right),$$

where the series

$$\sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=j+1}^{\infty} k(k-j)|\nu_{j,k}| \quad \text{and} \quad \sum_{j=1}^{\infty} j|\nu_{j,k}|$$

converge, and used this fact to show that the spectrum of H equals $[0, \infty)$. He also discussed the corresponding inverse spectral problem. This inverse spectral problem was subsequently

Date: August 11, 2003.

considered by Pastur and Tkachenko [8] for 2π -periodic potentials in $L^2_{\text{loc}}(\mathbb{R})$ of the form $\sum_{k=1}^{\infty} a_k e^{ikx}$.

In this paper, we will provide an elementary proof of the following result.

Theorem 2. *Let $V(x) = \sum_{k=1}^{\infty} a_k e^{ikx}$ with $\{a_k\}_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$. Then*

$$\Delta(V; z) = 2 \cos(2\pi\sqrt{z}),$$

where $\Delta(V; z)$ denotes the Floquet discriminant associated with (1) (cf. equation (2)).

Corollary 3. *Theorem 2 implies that the spectrum of H equals $[0, \infty)$; it also implies Theorem 1.*

Proof. In general, one-dimensional Schrödinger operators with periodic potentials have purely continuous spectra (cf. [9]). Since $-2 \leq 2 \cos(2\pi\sqrt{z}) \leq 2$ if and only if $z \in [0, \infty)$, one concludes that the spectrum of H equals $[0, \infty)$ and that Theorem 1 holds (see Lemma 5 below). \square

Remark. We note that the potentials V in Theorem 1 are nonreal and hence H is non-self-adjoint in $L^2(\mathbb{R})$ except when $V = 0$. It is known that $V = 0$ is the only *real* periodic potential for which the spectrum of H equals $[0, \infty)$ (see [1]). However, if we allow the potential V to be complex-valued, Theorem 1 provides a family of complex-valued potentials such that spectra of the associated Hill operators equal $[0, \infty)$. From the point of view of inverse spectral theory this yields an interesting and significant nonuniqueness property of non-self-adjoint Hill operators in stark contrast to self-adjoint ones. For an explanation of this nonuniqueness property of non-self-adjoint Hill operators in terms of associated Dirichlet eigenvalues, we refer to [4, p. 113].

As a final remark we mention some related work of Guillemin and Uribe [5]. Consider the differential equation (1) on the interval $[0, 2\pi]$ with the periodic boundary conditions. It is shown in [5] that all potentials in Theorem 1 generate the same spectrum $\{n^2 : n = 0, 1, 2, \dots\}$, that is, $\Delta(V; n^2) = 2$ for all $n = 0, 1, 2, \dots$.

2. SOME KNOWN FACTS

In this section we recall some definitions and known results regarding (1).

For each $z \in \mathbb{C}$, there exists a fundamental system of solutions $c(V; z, x)$, $s(V; z, x)$ of (1) such that

$$\begin{aligned} c(V; z, 0) &= 1, & c'(V; z, 0) &= 0, \\ s(V; z, 0) &= 0, & s'(V; z, 0) &= 1, \end{aligned}$$

where we use $'$ for $\frac{\partial}{\partial x}$ throughout this note. The *Floquet discriminant* $\Delta(V; z)$ of (1) is then defined by

$$(2) \quad \Delta(V; z) = c(V; z, 2\pi) + s'(V; z, 2\pi).$$

The Floquet discriminant $\Delta(V; z)$ is an entire function of order $\frac{1}{2}$ with respect to z (see [10, Chap. 21]).

Lemma 4. *For every $z \in \mathbb{C}$ there exists a solution $\psi(z, \cdot) \neq 0$ of (1) and a number $\rho(z) \in \mathbb{C} \setminus \{0\}$ such that $\psi(z, x + 2\pi) = \rho(z)\psi(z, x)$ for all $x \in \mathbb{R}$. Moreover,*

$$(3) \quad \Delta(V; z) = \rho(z) + \frac{1}{\rho(z)}.$$

In particular, if $V = 0$, then $\Delta(0; z) = 2 \cos(2\pi\sqrt{z})$.

For obvious reasons one calls $\rho(z)$ the *Floquet multiplier* of equation (1).

Lemma 5. *Let H be the Hill operator associated with (1) and $z \in \mathbb{C}$. Then the following four assertions are equivalent:*

- (i) z lies in the spectrum of H .
- (ii) $\Delta(V; z)$ is real and $|\Delta(V; z)| \leq 2$.
- (iii) $\rho(z) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$.
- (iv) Equation (1) has a non-trivial bounded solution $\psi(z, \cdot)$ on \mathbb{R} .

For proofs of Lemmas 4 and 5, see, for instance, [2, Chs. 1, 2, 5], [7], [9] (we note that V is permitted to be locally integrable on \mathbb{R}).

3. PROOF OF THEOREM 2

In this section we prove Theorem 2. In doing so, we will use the standard identity theorem in complex analysis which asserts that two analytic functions coinciding on an infinite set with an accumulation point in their common domain of analyticity, in fact coincide throughout that domain. Since both $\Delta(V; z)$ and $2 \cos(2\pi\sqrt{z})$ are entire functions, to prove that $\Delta(V; z) = 2 \cos(2\pi\sqrt{z})$, it thus suffices to show that $\Delta(V; 1/n^2) = 2 \cos(2\pi/n)$ for all integers $n \geq 3$.

Let $n \in \mathbb{N}$, $n \geq 3$ be fixed and let $\psi \neq 0$ be the solution of (1) such that $\psi(z, x + 2\pi) = \rho(z)\psi(z, x)$, $x \in \mathbb{R}$ for some $\rho(z) \in \mathbb{C}$. The existence of such ψ and ρ is guaranteed by Lemma 4. We set $\phi(z, x) = \psi(z, nx)$. Then

$$(4) \quad \phi(z, x + 2\pi) = \rho^n(z)\phi(z, x),$$

and

$$(5) \quad -\phi''(z, x) + q_n(x)\phi(z, x) = n^2 z \phi(z, x),$$

where

$$(6) \quad q_n(x) = n^2 V(nx) = n^2 \sum_{k=1}^{\infty} a_k e^{iknx},$$

with period 2π . Moreover, by (3) and (4),

$$(7) \quad \Delta(q_n; w) = \rho^n(z) + \frac{1}{\rho^n(z)}, \quad \text{where } w = n^2 z.$$

We will show below that

$$(8) \quad \Delta(q_n; 1) = 2 \quad \text{for every positive integer } n \geq 3.$$

First, if $w = 1$ (i.e., if $z = \frac{1}{n^2}$), then the fundamental system of solutions $c(q_n; 1, x)$ and $s(q_n; 1, x)$ of (5) satisfies

$$(9) \quad \begin{aligned} c(q_n; 1, x) &= \cos(x) + \int_0^x \sin(x-t) q_n(t) c(q_n; 1, t) dt, \\ s(q_n; 1, x) &= \sin(x) + \int_0^x \sin(x-t) q_n(t) s(q_n; 1, t) dt. \end{aligned}$$

Moreover, we have

$$(10) \quad s'(q_n; 1, x) = \cos(x) + \int_0^x \cos(x-t) q_n(t) s(q_n; 1, t) dt.$$

We use the Picard iterative method of solving the above integral equations. Define sequences $\{u_j(x)\}_{j \geq 0}$ and $\{v_j(x)\}_{j \geq 0}$ of functions as follows.

$$(11) \quad u_0(x) = \cos(x), \quad u_j(x) = \int_0^x \sin(x-t) q_n(t) u_{j-1}(t) dt,$$

$$(12) \quad v_0(x) = \sin(x), \quad v_j(x) = \int_0^x \sin(x-t) q_n(t) v_{j-1}(t) dt, \quad j \geq 1.$$

Then one verifies in a standard manner that

$$(13) \quad c(q_n; 1, x) = \sum_{j=0}^{\infty} u_j(x), \quad s(q_n; 1, x) = \sum_{j=0}^{\infty} v_j(x),$$

where the sums converge uniformly over $[0, 2\pi]$. Since

$$\Delta(q_n; 1) = c(q_n; 1, 2\pi) + s'(q_n; 1, 2\pi),$$

to prove that $\Delta(q_n; 1) = 2$, it suffices to show that the integrals in (9) and (10) vanish at $x = 2\pi$.

Next, we will rewrite (11) as

$$(14) \quad \begin{aligned} u_0(x) &= \frac{1}{2}(e^{ix} + e^{-ix}), \\ u_j(x) &= \frac{e^{ix}}{2i} \int_0^x e^{-it} q_n(t) u_{j-1}(t) dt - \frac{e^{-ix}}{2i} \int_0^x e^{it} q_n(t) u_{j-1}(t) dt, \quad j \geq 1. \end{aligned}$$

Using this and (6), one shows by induction on j that u_j , $j \geq 0$, is of the form

$$(15) \quad u_j(x) = \sum_{\ell=-1}^{\infty} b_{j,\ell} e^{i\ell x} \text{ for some } b_{j,\ell} \in \mathbb{C},$$

the sum converging uniformly for $x \in \mathbb{R}$. This follows from $n \geq 3$ because the smallest exponent of e^{it} that $q_n u_{j-1}$ can have in (14) equals 2. (The first three terms in (15) are due to the antiderivatives of $e^{\pm it} q_n(t) u_{j-1}(t)$, evaluated at $t = 0$.) Next we will use (13) and (15) to show that

$$(16) \quad \int_0^{2\pi} \sin(2\pi - t) q_n(t) c(q_n; 1, t) dt = 0.$$

We begin with

$$(17) \quad \begin{aligned} & \int_0^{2\pi} \sin(2\pi - t) q_n(t) c(q_n; 1, t) dt \\ &= -\frac{1}{2i} \int_0^{2\pi} (e^{it} - e^{-it}) q_n(t) c(q_n; 1, t) dt \\ &= -\frac{1}{2i} \int_0^{2\pi} (e^{it} - e^{-it}) \left(\sum_{k=1}^{\infty} a_k e^{iknt} \right) \left(\sum_{j=0}^{\infty} u_j(t) \right) dt \\ &= -\frac{1}{2i} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} a_k \int_0^{2\pi} (e^{i(kn+1)t} - e^{i(kn-1)t}) u_j(t) dt, \end{aligned}$$

where the change of the order of integration and summations is permitted due to the uniform convergence of the sums involved. The function $(e^{i(kn+1)t} - e^{i(kn-1)t}) u_j(t)$ is a power series in e^{it} with no constant term (cf. (15)), and hence its antiderivative is a periodic function of period 2π . Thus, every integral in (17) vanishes, and hence (16) holds. So from (9) we conclude that $c(q_n; 1, 2\pi) = 1$.

Similarly, one can show by induction that v_j for each $j \geq 0$ is of the form (15). Hence, from (10), one concludes that $s'(q_n; 1, 2\pi) = 1$ in close analogy to the proof of $c(q_n; 1, 2\pi) = 1$. Thus, (8) holds for each $n \geq 3$.

So by (7),

$$\Delta(q_n; 1) = \rho^n(1/n^2) + \frac{1}{\rho^n(1/n^2)} = 2 \text{ for every } n \geq 3.$$

This implies that $\rho^n(1/n^2) = 1$. So $\rho(1/n^2) \in \{\xi \in \mathbb{C} : \xi^n = 1\}$. Thus, $\Delta(V; 1/n^2) \in \{2 \cos(2k\pi/n) : k \in \mathbb{Z}\}$. Next, we will show that $\Delta(V; 1/n^2) = 2 \cos(2\pi/n)$.

We consider a family of potentials $q_\varepsilon(x) = \varepsilon V(x)$ for $0 \leq \varepsilon \leq 1$. For each $0 \leq \varepsilon \leq 1$, we apply the above argument to get that $\rho(\varepsilon, 1/n^2) \in \{\xi \in \mathbb{C} : \xi^n = 1\}$, where we use the notation $\rho(\varepsilon, 1/n^2)$ to indicate the possible ε -dependence of $\rho(1/n^2)$. Next, by the integral equations (9)–(12) with $q_\varepsilon = \varepsilon V$ instead of q_n , one sees that $\Delta(\varepsilon V; 1/n^2)$ can be written as a power series in ε that converges uniformly for $0 \leq \varepsilon \leq 1$. Thus, the function $\varepsilon \mapsto \Delta(\varepsilon V; 1/n^2) \in \{2 \cos(2k\pi/n) : k \in \mathbb{Z}\}$ is continuous for $0 \leq \varepsilon \leq 1$ (in fact, it is entire w.r.t.

ε). Since $\{2 \cos(2k\pi/n) : k \in \mathbb{Z}\}$ is discrete, and since $\Delta(\varepsilon V; 1/n^2) = 2 \cos(2\pi/n)$ for $\varepsilon = 0$, we conclude that

$$\Delta(\varepsilon V; 1/n^2) = \Delta(0; 1/n^2) = 2 \cos(2\pi/n) \text{ for all } 0 \leq \varepsilon \leq 1.$$

In particular, $\Delta(V; 1/n^2) = 2 \cos(2\pi/n)$ for every positive integer $n \geq 3$. Since $\Delta(V; z)$ and $2 \cos(2\pi\sqrt{z})$ are both entire and since they coincide at $z = 1/n^2$, $n \geq 3$, we conclude that

$$\Delta(V; z) = 2 \cos(2\pi\sqrt{z}) \text{ for all } z \in \mathbb{C}$$

by the identity theorem for analytic functions alluded to at the beginning of this section. This completes proof of Theorem 2 and hence that of Theorem 1 by Corollary 3.

Remarks. (i) Adding a constant term to the potential V yields a translation of the spectrum. (ii) If the potential V is a power series in e^{-ix} with no constant term, then the spectrum of H is still $[0, \infty)$, by complex conjugation. (iii) If V lies in the $L^2([0, 2\pi])$ -span of $\{e^{ikx}\}_{k \in \mathbb{N}}$, then by continuity of $V \mapsto \Delta(V; z)$ one concludes $\Delta(V; z) = 2 \cos(2\pi\sqrt{z})$ and hence the spectrum of H equals $[0, \infty)$ (see [8]). (iv) Potentials V that include negative and positive integer powers of e^{ix} are not included in our note. Consider, for example, equation (1) with $V(x) = 2 \cos(x)$, the so-called Mathieu equation. The spectrum of H in this case is known to be a disjoint union of infinitely many closed intervals on the real line [6] (also, see [2], [7]). In particular, the spectrum of H is not $[0, \infty)$. In such a case the antiderivatives of $(e^{i(kn+1)t} - e^{i(kn-1)t})u_j(t)$ in (17) need not be periodic and our proof breaks down.

Acknowledgments. The author thanks Fritz Gesztesy and Richard Laugesen for suggestions and discussions to improve the presentation of this note.

REFERENCES

- [1] G. Borg, Eine Umkehrung der Sturm-Liouvillschen Eigenwertaufgabe. *Acta Math.*, 78: 1–96, 1946.
- [2] M. S. P. Eastham, *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, London, 1973.
- [3] M. G. Gasymov, Spectral analysis of a class of second-order non-self-adjoint differential operators, *Funct. Anal. Appl.*, 14: 11–15, 1980.
- [4] F. Gesztesy and H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions. Vol. I: (1 + 1)-Dimensional Continuous Models*, Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge Univ. Press, 2003.
- [5] V. Guillemin and A. Uribe, Hardy functions and the inverse spectral method, *Comm. P. D. E.*, 8: 1455–1474, 1983.
- [6] E. N. Ince, A proof of the impossibility of the coexistence of two Mathieu functions, *Proc. Camb. Phil. Soc.*, 21: 117–120, 1922.
- [7] W. Magnus and S. Winkler, *Hill's Equation*, Dover Publications, Inc., New York, 1979.
- [8] L. A. Pastur and V. A. Tkachenko, Spectral theory of Schrödinger operators with periodic complex-valued potentials, *Funct. Anal. Appl.*, 22: 156–158, 1988.
- [9] F. S. Roze-Beketov, The spectrum of non-selfadjoint differential operators with periodic coefficients, *Sov. Math. Dokl.*, 4: 1563–1566, 1963.
- [10] E. C. Titchmarsh, *Eigenfunction Expansions associated with Second-Order Differential Equations, Part II*, Oxford University Press, New York, 1958.

E-MAIL: kcshin@math.missouri.edu

DEPARTMENT OF MATHEMATICS , UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA